

Fibonacci Numbers **ANSWERS**

Lesson 1 of 10, work individually or in pairs

In 1202, the mathematician Leonardo Pisano Fibonacci (pronounced fi-buh-NAH-chee) published a book with the famous Fibonacci sequence in it. (A sequence is a list of numbers.)

This sequence starts with the numbers 1 and 1.

To get the next number, you add the first two:
 $1 + 1 = 2$.

Now the sequence looks like 1, 1, 2.

To get the next number, you add the previous two numbers: $1 + 2 = 3$.

Now the sequence looks like 1, 1, 2, 3.

Keep doing this to build the sequence. The next two numbers in the sequence are 5 (from $2 + 3 = 5$) and 8 (from $3 + 5 = 8$).

Now the sequence looks like 1, 1, 2, 3, 5, 8.

*Materials Needed: Pencil
Math Journal or Notebook*

1. Write down the first part of the sequence, shown below, and then fill in the blanks for the next 6 Fibonacci numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

Draw a box around this list of numbers so you can find it easily. You will use it for several different activities.

2. In the first few terms (numbers) of the sequence, you can see a pattern of whether the numbers are even or odd. Make a table to see the pattern. Make as many rows as you need so you have a row for each number.

Number	Even or odd
1	odd
1	odd
2	even
3	odd
5	odd
8	even
13	odd
21	odd
34	even
55	odd
89	odd
144	even

3. Write 1-2 sentences to describe the pattern and explain why it happens. (Hint: What is an odd number plus an odd number?)

The pattern is odd, odd, even, odd, odd, even. This happens because if you add two odds, you'll get an even number. For the next two steps, if you add an odd and an even you get an odd number.

Standards: Patterns, numbers, reasoning, communication

Fibonacci Numbers **ANSWERS**

Lesson 2 of 10, work in pairs

There are many patterns in the Fibonacci numbers. Open your notebook to look at the list of Fibonacci numbers that you made in Lesson 1.

Make sure that they are

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and so on.

One pattern can be seen when you add up Fibonacci numbers.

1. Write the sums below in your math journal, along with the answers that go in the blanks.

$$1 + 1 = \underline{2}$$

$$1 + 1 + 2 = \underline{4}$$

$$1 + 1 + 2 + 3 = \underline{7}$$

$$1 + 1 + 2 + 3 + 5 = \underline{12}$$

$$1 + 1 + 2 + 3 + 5 + 8 = \underline{20}$$

*Materials Needed: Pencil
Math Journal or Notebook*

2. In the answers to each sum in problem 1, you don't see the Fibonacci numbers. You can see numbers that are close to the Fibonacci numbers. How do the answers relate to the Fibonacci numbers?

The answers are one less than the Fibonacci numbers.

3. **Predict** What is the next step in the pattern of additions? What answer do you expect?

The next addition is $1 + 1 + 2 + 3 + 5 + 8 + 13$ and you expect the answer to be 33, which is one less than 34. It is.

4. **Check** Check to see if your prediction is correct by completing the next several steps in the pattern.

$$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 = 54$$

$$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 = 88$$

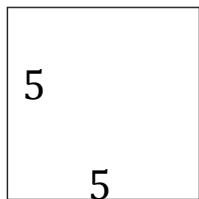
Standards: Patterns, number theory, reasoning

Fibonacci Numbers **ANSWERS**

Lesson 3 of 10, work in pairs

Review of exponents: An exponent tells you how many times to multiply a number by itself. For example, $5^2 = 5 \times 5$ and $5^3 = 5 \times 5 \times 5$.

You can also see an exponent of 2 as giving the area of a square: $5^2 = 5 \times 5$ shows the area of the square of side length 5.



Now we will use exponents to see another pattern in the Fibonacci numbers.

Remember that the Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, and so on.

*Materials Needed: Pencil
Math Journal or Notebook
Calculator (optional)*

1. Write the sums below in your math journal, along with the answers that go in the blanks.

$1^2 + 1^2 = \underline{2}$ *When you have $1^2 + 1^2$, do the exponents first, so the answer is $1 + 1 = 2$.*

$1^2 + 2^2 = \underline{5}$

$2^2 + 3^2 = \underline{13}$

$3^2 + 5^2 = \underline{34}$

$5^2 + 8^2 = \underline{89}$

2. What is the pattern of the answers you got? (Are the answers related to the Fibonacci numbers?) *Yes, the answers are Fibonacci numbers—but you skip some in between.*

3. **Predict** What is the step in the pattern of exponents and additions? What answer do you expect? You may have to add a few more Fibonacci numbers to your list to get the next step.

The next equation is $8^2 + 13^2$, and you expect it to be 233 because the Fibonacci numbers are being skipped. When we check, we see that $8^2 + 13^2 = 64 + 169 = 233$.

4. **Check** Check to see if your prediction correct by completing the next several steps in the pattern.

$13^2 + 21^2 = 610$ and $21^2 + 34^2 = 1597$. The next Fibonacci numbers not listed at the left are 987 and 1597, so the pattern seems to continue.

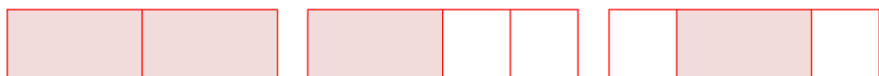
Standards: Patterns, number theory, reasoning

Fibonacci Numbers **ANSWERS**

Lesson 4 of 10, work in pairs

(Text of left hand side deleted to make room for answers.)

1. Now figure out how many ways there are to cover the row of 4 squares. Record all of the different pictures in your notebook.



2. Repeat with 5 squares.



Materials Needed: Pencil
Math Journal or Notebook

Master to go with Fibonacci Numbers Lesson 4

3. Make a table like the one shown below in your notebook.

Row of ___ squares	Number of ways to cover with squares and rectangles
3	3
4	5
5	8

4. **Predict** Look back at the list of Fibonacci numbers you made on Lesson 1 and compare it to the numbers in the table above. What do you think will be the next row in the table?

The correct prediction is that it'll be 13. A common incorrect prediction is that it will be 12. Allow students to have that prediction and discover the right answer in #5.

5. **Check** Use the row of 6 squares to see if your prediction is correct.



6. **Challenge Question** Can you find all of the ways to cover a row of 7 squares with these small squares and rectangles? *This is tough—there are 21 of them! Students need to work methodically to list them all.*

Standards: Patterns, combinations, reasoning

Fibonacci Numbers **ANSWERS**

Lesson 5 of 10, work in pairs

Materials Needed: Pencil
Math Journal or Notebook

Remember that the Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and so on.

Every counting number can be written as the sum of Fibonacci numbers, where no number is used more than once.

These examples show how this works.

$$9 = 8 + 1$$

$$25 = 21 + 3 + 1$$

$$100 = 89 + 8 + 3$$

1. Write the numbers below as a sum of Fibonacci number (without using any number twice)

$$55 = 34 + 21 \text{ (there may be other answers)}$$

$$75 = 55 + 13 + 5 + 2 \text{ (there may be other answers)}$$

$$200 = 144 + 55 + 1 \text{ (there may be other answers)}$$

2. Is there more than one way to write some numbers as a sum of Fibonacci numbers? For example, we could write $9 = 8 + 1$ or instead as $9 = 5 + 3 + 1$. What other numbers can you write in more than one way?

There are multiple answers. Check to make sure no Fibonacci number was used twice.

3. **Explain** Do you find a method that would always work for picking the right Fibonacci numbers to add up to the numbers? Write one or two sentences to explain how you could find the numbers to add. *One possible answer: Try to start with the largest possible Fibonacci number.*

Standards: Patterns, number theory, communication

Fibonacci Numbers **ANSWERS**

Lesson 6 of 10, work in pairs

Remember that the Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and so on.

Now let's look at another pattern in the Fibonacci numbers.

1. Find the quotients, using long division or a calculator. Record the answer up through the thousandths place. Record the problem and the answer in your math journal.

$$1 \div 1 = \underline{1}$$

$$2 \div 1 = \underline{2}$$

$$3 \div 2 = \underline{1.5}$$

$$5 \div 3 = \underline{1.667}$$

$$8 \div 5 = \underline{1.6}$$

*Materials Needed: Pencil
Math Journal or Notebook
Calculator (optional)*

2. Look at the list of Fibonacci numbers to the left, and at the pattern of divisions done in problem 1. Figure out the pattern, and fill in the next 6 pieces of the pattern. Record your answers in a column as shown below.

$$13 \div 8 = 1.625$$

$$21 \div 13 = \underline{1.615}$$

$$34 \div 21 = \underline{1.619}$$

$$55 \div 34 = 1.618$$

$$89 \div 55 = 1.618$$

$$144 \div 89 = 1.618$$

3. **Examine** Now look at the list of answers you got for problems 1 and 2. Are the answers getting close to some number? Write down your ideas about the answer, and you'll use them in Lesson 7. *They're getting closer to 1.618.*

Standards: Patterns, division, reasoning

Fibonacci Numbers and the Golden Ratio **ANSWERS**

Lesson 7 of 10, work in pairs

Look at the answer to Question 3 of Lesson 6 before you read further.

If you keep doing divisions of consecutive (one right after the other) Fibonacci numbers, like you did in Lesson 6, you will get closer and closer to a special number called the **golden ratio**. The golden ratio shows up in art, architecture, music, and nature. The ancient Greeks thought that rectangles whose sides form a golden ratio were pleasing to look at.

The golden ratio is often symbolized by the Greek letter phi, which looks like this: ϕ . It is the number $\phi = 1.6180339887\dots$ (and so on). When you write out the decimal, it continues on forever without repeating or making a pattern.

*Materials Needed: Pencil
Math Journal or Notebook
Calculator (optional)
Boxes or cans*

1. Collect boxes and cans from home, or use some that are in the classroom. For boxes, measure the height and width of the boxes, and for cans, measure the height and diameter. Make a table of what you measured.

Description	Height in cm	Width in cm	longer one \div shorter one
<i>Cereal box</i>	<i>30.0</i>	<i>19.4</i>	<i>1.546</i>
<i>Answers will vary</i>			

2. **Examine** Is the quotient close to the golden ratio ϕ ? It probably will never exactly equal ϕ , but many of the items are probably close. Which ones are closest?

3. **Explain** Why do you think that the common household items show the golden ratio?

4. **Independent Research** Do an internet search on “golden ratio architecture,” “golden ratio art,” or “golden ratio nature” and see what else you can learn about the golden ratio.

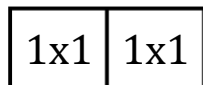
Standards: Measurement, data collection, communication

Fibonacci Numbers **ANSWERS**

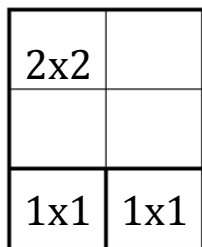
Lesson 8 of 10, work in pairs

Remember that the Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and so on.

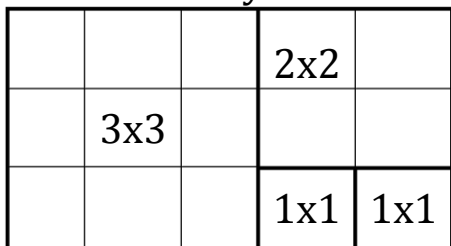
The Fibonacci numbers can be visualized as a spiral of squares. First make a 1x1 square. Then make a square that shares one side of the first and is also 1x1.



Then make a new square above the 1x1 squares that is 2x2.

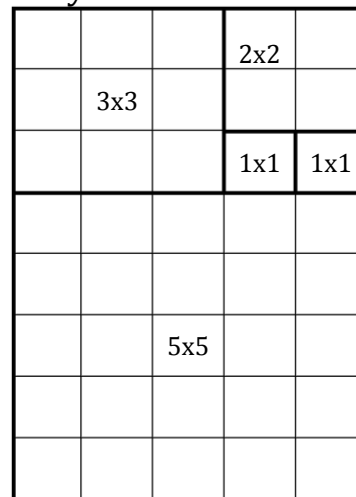


Then make a new square to the left of the ones you've made—it has to be 3x3 if it shares a whole side with what you've already drawn.



*Materials Needed: Pencil
Math Journal or Notebook
Graph Paper*

Then make a new square below what you've already made. It must be 5x5 to share a side with the figure you already have.



1. Continue this pattern on a piece of graph paper (the next square will be to the right, then above, then to the left, then below, and so on). Make sure you start near the middle of the paper to make room.

See teacher's guide or have students compare answers with an online image: Go to Google Images (images.google.com) and search "Fibonacci rectangle."

2. Look again at the ratios you computed in Lesson 7. Now think about the rectangles in all of the in-between steps (such as the rectangles shown on this card). Write down the ratio between the long side and the short side of each in-between step. What do you notice?

These ratios are the same as the ones in lesson 7.

Standards: Patterns, number theory, reasoning

Fibonacci Numbers **ANSWERS**

Lesson 9 of 10, work in pairs

François Édouard Anatole Lucas (pronounced loo KAH) studied mathematics in the 1800s. He came up with a new number sequence, which are called the Lucas numbers.

The Lucas numbers are made in just the same way as the Fibonacci numbers, except the first two numbers are 2 and 1.

If you don't remember how to make the Fibonacci numbers, look back at Lesson 1.

The Lucas are:

2
1
3
4
7
11
18
29

and so on.

Materials Needed:
Math Journal or Notebook
Calculator (optional)

1. **Compare** Redo some of the examples in Lessons 1-6, except now use the Lucas numbers instead of the Fibonacci numbers. Carefully record your results in your math journal. Some questions you could ask are:

- Do Lucas numbers follow the odd, odd, even pattern, or something similar? *yes*
- Does the golden ratio show up when you take quotients of consecutive (one right after the other) Lucas numbers? *yes*
- Can you write any counting number as a sum of Lucas numbers if you do not repeat any of the Lucas numbers? *probably*
- Do other number patterns show up in the Lucas numbers? *Answers will vary.*

As you do this exercise, you may find out that the answer to some of the questions is, "No, there isn't a pattern." Learning that there is not the same pattern in the Lucas numbers is also interesting—mathematicians like to learn when things are different as well as when they are the same. So, when you find that there isn't a pattern, record that in your math journal also!

Standards: Patterns, number theory, reasoning

Fibonacci Numbers **ANSWERS**

Lesson 10 of 10, work individually or in pairs

Create This is your chance to be creative with mathematics! Create your own sequence of numbers. Perhaps you will use the same rule as the Fibonacci and Lucas numbers. Perhaps you will use a different rule to create the sequence.

1. Write down the rule to create your sequence in your notebook. Also write down the first 10-12 numbers in the sequence. Give your numbers a name!

2. **Compare** Look again at the patterns you learned in Lessons 1-6. Does your number sequence have some of the same patterns? Are there other patterns in your number sequence? Carefully write down which patterns you have. Also write down which patterns you looked for that your number sequence does not have.

*Materials Needed: Pencil
Math Journal or Notebook
Internet or library for research*

3. Tell one of your classmates about your number sequence, and learn about the number sequence your classmate created.

4. **On Your Own** Prepare a presentation about your number sequence and the patterns in it. Make a slide show in PowerPoint or make a poster to share with your class.

5. **Independent Research** Do some research on the internet to find out more about the Fibonacci numbers. Read about Leonardo Fibonacci or François Édouard Anatole Lucas. Write a report or prepare a presentation.

Skills: Patterns, number theory, reasoning, communication