The graphs we will explore in this unit have the same name as the graphs you usually draw in math class, but have a very different and particular meaning.
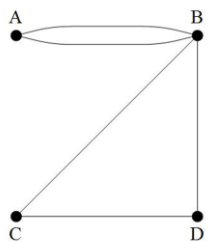
A graph is just a collection of dots called vertices (just one dot is a vertex), and lines connecting them called edges. This graph has 4 vertices and 5 edges.
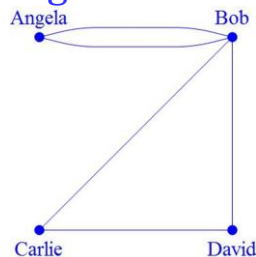
Graphs can be drawn to represent many situations. Example: During one week, Angela, Bob, Carlie, and David played games.

    Angela and Bob played War
    Bob, Carlie, and David played Go Fish
    Angela and Bob played Mancala

We can draw a graph where the vertices represent the people, and there is an edge between two people if they played a game together.

Since Angela and Bob played two games together, there are two edges between them. The graph would look like the one shown here.

1. In one week in the summer, a 5 different youth baseball teams had games. The game schedule was:
Rockies vs. Yankees   Athletics vs. Dodgers
Red Sox vs. Rockies   Athletics vs. Yankees
Dodgers vs. Red Sox

   a. Draw a graph representing the week's games, where the vertices are the teams and there is an edge between two teams if they played a game together.

   b. Write down how many vertices and edges your graph has.

2. A preschool teacher wrote down which children played with each other during a day.

        Anita played with Carlos, Paige, and Anna.
        Tomas played with Leon.
   Carlos played with Tomas.
        Paige played with Anna and Leon.

   a. Make a graph that shows the relationships, where the children are represented by the vertices and there is an edge between two children if they played together.

   b. Write down how many vertices and edges your graph has.

3. Compare the graph you made with one your classmate made. Do they look exactly the same?
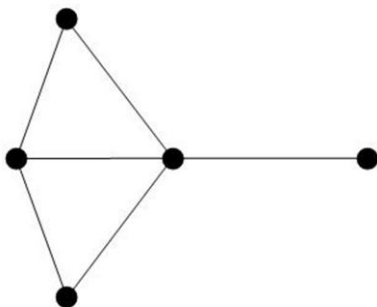
Materials Needed: Pencil
*Math Journal or Notebook*
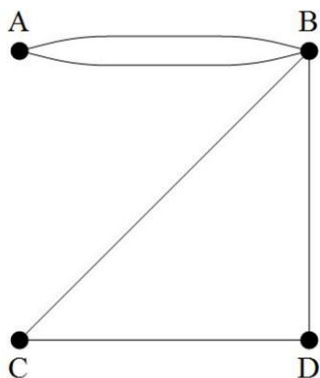
The degree of a vertex is the number of edges coming from that vertex. In the graph above, vertex A, vertex C, and vertex D all have degree 2, and vertex B has degree 4.

1. Find the degree of every vertex of this graph.

2. Find the degree of every vertex of this graph. When two lines cross, but there isn't a dot, that does not count as a vertex, so this graph has only 6 vertices.
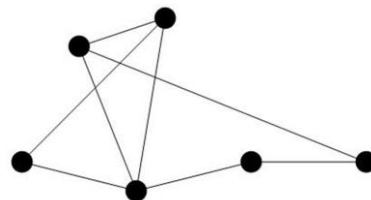
3. For the three graphs on this page (the example problem and problems 1 and 2), fill in the table.

| Graph	Sum of the degrees of the vertices	number of edges
example problem	$2 + 4 + 2 + 2 = 10$	
Problem 1		
Problem 2		

4. **Predict** Look at the table in problem 3. Do you see a relationship between the sum of the degrees of the vertices and the number of edges? Explain that relationship.

5. **Check** Make your own graph and see if the relationship you found in problem 5 works for that graph.

Standards: Representation, patterns

Materials Needed: Pencil
Math Journal or Notebook
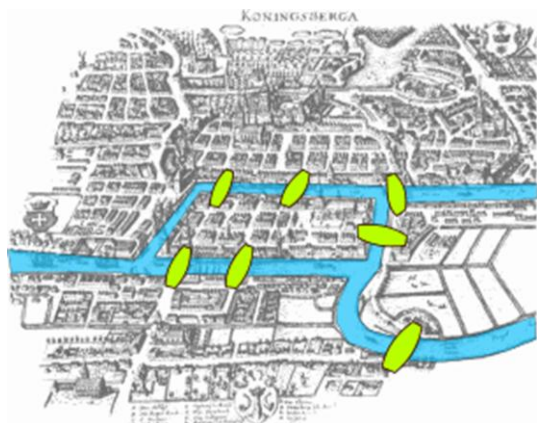
In the 1700s, there was a city named Koenigsburg in East Prussia (it is now Kaliningrad and is in Russia).

The Pregel River runs through Koenigsburg, and it is split between two branches. There were 7 bridges in the city, connecting the riverbanks and the islands, as shown above.

Bridges of Koenigsburg problem Is it possible for the person to take a walk through the city, crossing each bridge exactly once, and finishing at the starting point?
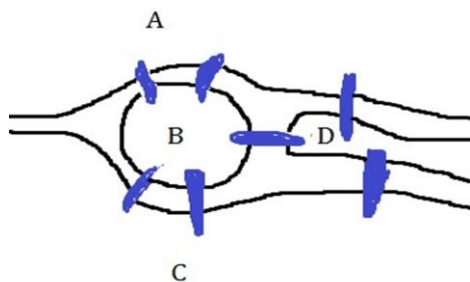

A simplified picture of the city.

1. Try to solve this problem. Write down your experiments in your notebook. It may be helpful to use the simplified picture of the city shown lower left. Do you think that a person can walk through the city, crossing every bridge exactly 1 time, and start and end at the same point?

The mathematician Leonhard Euler (pronounced "oiler") answered this question with graph theory! He drew a graph with the regions A, B, C, and D as vertices and the bridges as edges. It looks like this.
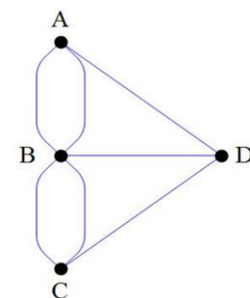
2. Find the degree of every vertex of the graph that represents Koenigsburg.

Any path that traces the graph by going across each edge exactly once and starts and ends at the same point is called an **Euler circuit**.

3. Explain why find an answer to the Bridges of Koenigsburg problem is the same as finding an Euler circuit on the graph shown.

*Standards: Geometry, representation*

Look back at the Bridges of Koenigsburg problem and the definition of an Euler circuit in Lesson 3.

Euler showed that if there is any vertex of the graph that has odd degree, then the graph does not have an Euler circuit. He also showed that if all of the vertices of the edges have an even degree, then the graph does have an Euler circuit.

This graph has an Euler circuit, because all of the vertices have even degree. You can find the circuit by traveling along the graph in this order:
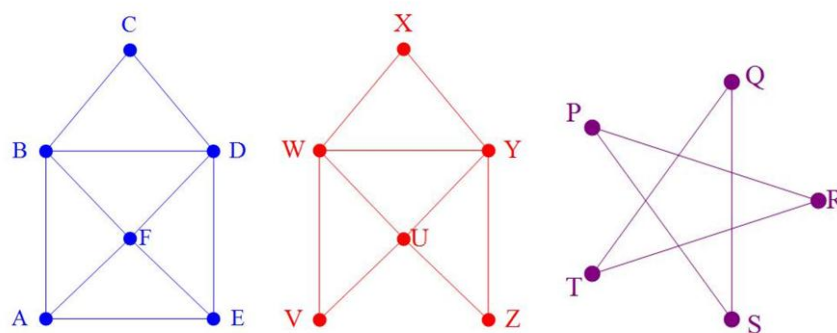A → B →D →C →B →A.
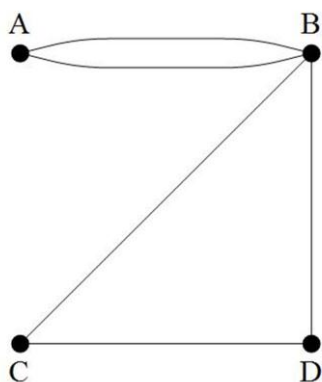
(Notice that vertex B is used twice. This is fine in an Euler circuit.)

1. Does the graph showing the Koenigsburg bridges have an Euler circuit? Why or why not?

2. For each of the graphs shown below, decide if they have an Euler circuit. If they do, find an Euler circuit. If it doesn't have an Euler circuit, explain why not.

3. For the graphs in problem 2 that have Euler circuits, compare your Euler circuit to a classmate's Euler circuit. Do you have the same circuits?

Standards: Representation, reasoning

Materials Needed: Pencil
Math Journal or Notebook

Another problem that can be studied through graph theory is what is called the "traveling salesman problem."   One example from that problem is shown below. It shows 4 cities that a sales representative would visit, and the distances (in miles) between the cities. The problem is to find the best order for the sales representative to visit the cities, and return to the starting point, so that the shortest distance possible is travelled.
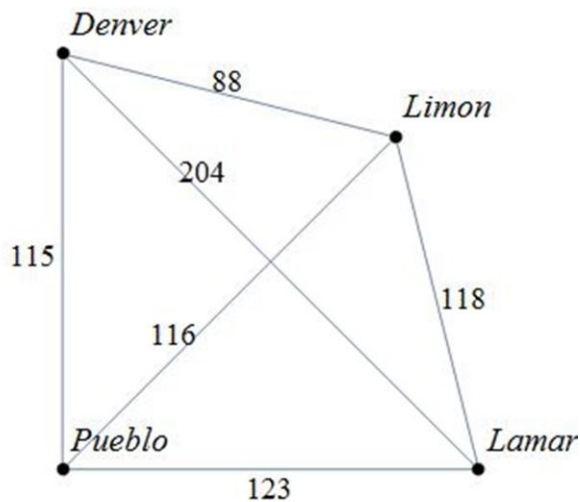
Now the graphs have a label on the edges. This is called a *weighted graph*.

1. Write down some possibilities for paths the sales representative could take, starting in Denver. Which one is the shortest? Write all of the possibilities that you tested in your math journal.

2. A family that lives in Colorado Springs wants to visit some of Utah's national parks. Distances between locations are shown in the table below. Make a directed graph, with the city and parks as the vertices, and the distances shown on each edge.

	Colorado Springs	Arches National Park	Bryce Canyon National Park
Arches	408 miles		
Bryce	617 miles	267 miles	
Zion	681 miles	335 miles	88 miles

3. Find the path that the family should take, starting and ending in Colorado Springs, to travel the shortest distance.

Standards: Representation, combinations

Materials Needed: Pencil
Math Journal or Notebook

A traveling salesman problem, like the ones you did in Lesson 5, has a special kind of graph where every vertex is connected to every other vertex by an edge. This is called a *complete graph*.

We are going to count the number of paths to check to solve a traveling salesman problem in a complete graph. Let's start with the easiest one, the complete graph with 3 vertices.
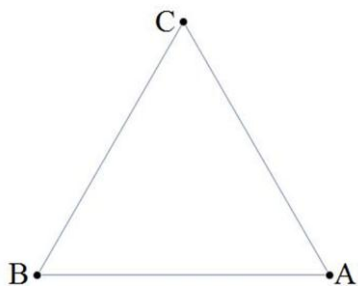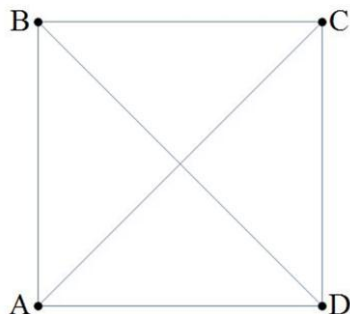
If we want to list all possible paths that start and end at the vertex A, we get only two:
A→B→C→A
A→C→B→A

1. Now look at the complete graph with 4 vertices. List all of the paths that start and end at A.

Look at the list you made in problem 1. Notice that if you are at vertex A, you have 3 choices of where to go. Then you had 2 choices of where to go next. Once you get to the third place, your path is already determined—you can't go to A, so you must go to the remaining vertex.

2. Use logic to figure out how many paths there are, starting and ending at vertex A, on a complete graph with 5 vertices. You should not list all possible paths, but instead, try to figure out an answer with logic.
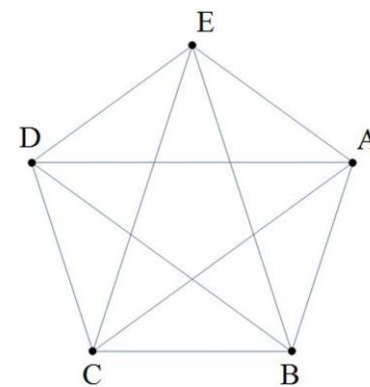
3. **Challenge Question** How do your answers change if it doesn't matter whether you travel a path forward or backward (if A→B→C→A and A→C→B→A are the same path)?

Standards: Representation, patterns, combinations

Materials Needed: Pencil and highlighter
Math Journal or Notebook

In this lesson, you will find out how many edges there are in a complete graph (one where every vertex is connected to every other vertex by an edge).

1. Draw this complete graph with 5 vertices your math journal.
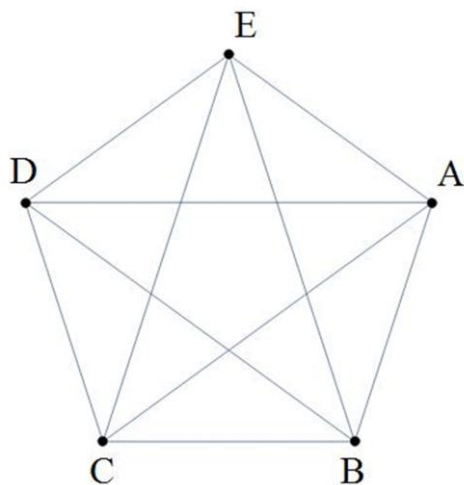
2. If you look at vertex A, how many edges come out of that vertex? Color each of those edges with a highlighter or marker, and write down how many there are in your math journal.

3. Now look at vertex B. How many more edges (ones that are not colored yet) come from vertex B? Color those edges, and write down how many there are.

4. Now look at vertex C. How many more edges (ones that are not colored yet) come from vertex C? Color those edges, and write down how many there are.

5. Now look at vertex D. How many more edges (ones that are not colored yet) come from vertex D? Color those edges, and write down how many there are. Now all of your edges should be colored.

6. Add up the answers to questions 2, 3, 4, and 5 to find out how many edges are in the graph. Check your answer by counting edges.

7. **Challenge Question** What if the graph had 6 vertices, and all of them were connected together? How many edges does that graph have? Draw it and check your answer.

Standards: Representation, patterns