Look at this sequence of triangles.

Each triangle is an equilateral triangle. (The first one doesn’t look like a triangle, but it would be the triangle that you could make if every side has only one dot.) The second triangle has 2 dots on each side, the third has 3 dots on each side, and so on. If we want to count the number of dots in each triangle, the first one has 1 dot, the second has $1 + 2 = 3$ dots, the third one has $1 + 2 + 3 = 6$ dots, and the fourth one has $1 + 2 + 3 + 4 = 10$ dots. Because the numbers 1, 3, 6, and 10 come from counting dots in a triangle, they are called **triangular numbers**.

If we want to find the next triangular number, we could draw another triangle like this and count the dots

or we could just imagine that the next triangle has

$1 + 2 + 3 + 4 + 5 = 15$ dots.

1. Find the next five triangular numbers. In your math journal, write the triangular numbers as a list of numbers:

   \[ 1, 3, 6, 10, 15, \underline{21}, \underline{28}, \underline{36}, \underline{45}, \underline{55} \]

   Circle this list of numbers—you are going to use it in several lessons.

2. The first triangular number is 1. The 2\textsuperscript{nd} triangular number is $1 + 2$. The 3\textsuperscript{rd} triangular number is $1 + 2 + 3$ and the 4\textsuperscript{th} is $1 + 2 + 3 + 4$. What numbers would you add to find the 20\textsuperscript{th} triangular number?  \textit{(Don’t add it up, just write down the addition problem.)}

   \[ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 \]

3. Look at the list of list of triangular numbers in problem 1. They make the pattern odd, odd, even, even, odd, odd, even, even, and so on. Do you think this same pattern continues? Why or why not?  \textit{Answers will vary. (The pattern does continue, but students may or may not know that.)}

**Materials Needed:** Pencil

Math Journal or Notebook

**Standards:** Patterns, representation
1. Look at question 2 of Lesson 1, where you wrote the sum for the 20th triangular number.
   a. Match up the numbers the way Gauss did. What is the sum of each match? The sum of each match is 21 (20+1, 19+2, etc).
   b. How many matches (pairs) of numbers do you have? There are 10 pairs.
   c. Multiply your answers from parts a and b to find the value of the 20th triangular number. 21 × 10 = 210

2. Try to use Gauss’ method to find the 7th triangular number, 1 + 2 + 3 + 4 + 5 + 6 + 7. It doesn’t quite work, but you can modify the method so that it does. Once you find the sum, make sure it matches a number in your list from problem 1 on Lesson 1. Can do 8 + 8 + 8 + 4 = 28 or (8 × 3) + 4 = 28 or any other correct method.

3. Use this method to find the 1,000th triangular number. There are 500 pairs, and they each add to 1001. So do 1001 × 500 = 500,500.

4. How would you use a similar method to find the sum 2 + 4 + 6 + 8 + ⋯ + 198 + 200? Be careful when you count the number of pairs of numbers. (Hint: Figure out how many numbers are added in 2 + 4 + 6 + 8 + ⋯ + 198 + 200.) There are only 50 pairs, and they each add to 202. Compute 202 × 50 = 10,100.
Another way to find triangular numbers is to think about the triangles being right triangles instead of equilateral triangles.

Now we can put together two of the same triangular numbers to make a rectangle, like this:

The rectangle with 2 of the triangular numbers had $3 \times 2 = 6$ dots, so each of the triangular numbers is $6 \div 2 = 3$ dots.

If we do the same thing we the third triangular number, we see that two of them together have $4 \times 3 = 12$ dots, so the third triangular number is $12 \div 2 = 6$.

If we put it all together, we see that the $2^{nd}$ triangular number is $3 \times 2 \div 2 = 3$
the $3^{rd}$ triangular number is $4 \times 3 \div 2 = 6$
the $4^{th}$ triangular number is $5 \times 4 \div 2 = 10$

1. Do you see the pattern in red? Use that pattern to find the $100^{th}$ triangular number. Write the answer in your notebook.

   The $100^{th}$ triangular number is $102 \times 101 \div 2 = 5,050$.

2. **Check** Compare your answer to problem 1 with the story about Gauss in Lesson 2. Did you get the same answer? If you didn’t, go back and check your work.

   **It is the same.**

3. **Challenge Question** Why does looking at the rectangles do the same kind of matching that Gauss did? Write one or two sentences to explain why the picture shows Gauss’ matching method.

   **In the picture at the right, we’re making a rectangle where the top row is the smallest number + the largest number. The next row is the $2^{nd}$ smallest number + the $2^{nd}$ largest number, and so on.**

**Materials Needed:** Pencil, Math Journal or Notebook

**Standards:** Numbers, geometry, patterns
Now let’s look at square numbers.

1. How many dots are in each square? 
   1, 4, 9, 16

2. What number do you multiply by itself to get the number of dots in each square? 
   1 × 1, 2 × 2, 3 × 3, 4 × 4

3. What is the next square number? 
   5 × 5 = 25

4. The first square number is 1 × 1 = 1. The 2nd square number is 2 × 2 = 4. The 3rd square number is 3 × 3 = 9. What is the 20th square number? 
   20 × 20 = 400

5. Now look at the differences between the square numbers. How many dots were added to the first square to get the second square? How many dots were added to the second square to get the third square? Make a table like this in your notebook:

<table>
<thead>
<tr>
<th>To get from the 1st to the 2nd, add</th>
<th>3 dots</th>
</tr>
</thead>
<tbody>
<tr>
<td>to get from the 2nd to the 3rd, add</td>
<td>5 dots</td>
</tr>
<tr>
<td>to get from the 3rd to the 4th, add</td>
<td>7 dots</td>
</tr>
<tr>
<td>to get from the 4th to the 5th, add</td>
<td>9 dots</td>
</tr>
</tbody>
</table>

6. **Communicate** Look at the table above and write the pattern that describes what happens from one row to the next row. Write a sentence or two to describe that pattern.
   *Answers will vary. Two correct ones are:* To get to the next square number, add 2 more dots than you did for the previous square number. OR: Always add an odd number of dots.

**Standards:** Numbers, geometry, patterns
So far we have computed **triangular numbers** 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, and so on. We also computed the **square numbers** 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, and so on.

1. Find out what happens when you add consecutive triangular numbers. (Consecutive means one right after the other.) List the sums neatly in your notebook, as shown. Do this for all numbers on the list, so you will have 9 different sums listed.

   \[
   
   \begin{align*}
   1 + 3 &= 4 \\
   3 + 6 &= 9 \\
   6 + 10 &= 16 \\
   10 + 15 &= 25 \\
   15 + 21 &= 36 \\
   21 + 28 &= 49 \\
   28 + 36 &= 64 \\
   36 + 45 &= 81 \\
   45 + 55 &= 100
   \end{align*}
   
   \]

2. **Connect** Now look at the answers to sums of consecutive triangular numbers. Do you see any connection between the answers and one of the lists of numbers? *The sum of two consecutive triangular numbers is a square number. Neat!*

   Look at the two pictures below. They show how we can have the sums 1 + 3 = 4 and 3 + 6 = 9 in pictures.

   ![Picture 1](image1.png)
   ![Picture 2](image2.png)

3. Using two different colors, draw a picture like the ones above representing the sum 6 + 10 = 16.

   ![Picture 3](image3.png)

**Standards:** Numbers, geometry, patterns
You don’t have to do just triangular numbers or square numbers! You can also do numbers that are in the shape of a pentagon or hexagon or any figure.

1. The first three pentagonal numbers are 1, 5, and 12. They are shown below. What is the next one?

The next one is 22.

2. Every pentagonal number can be written as a sum of a triangular number and a square number. For example, 12 is a pentagonal number, \(3 + 9 = 12\). 3 is a triangular number and 9 is a square number. 51 is a pentagonal number. Which triangular number and a square number do you have to choose so that they add to 51?

\[15 + 36 = 51\]

3. Extend Choose another shape to make numbers from. Describe it in words and pictures. List the first several numbers in your number sequence. See if you can find any patterns or a general way of finding the next number in your sequence.

Standards: Numbers, patterns

Materials Needed: Pencil Math Journal or Notebook